Math 612 – Algebraic Topology II

Note Title 8/25/2014

Administrative details:

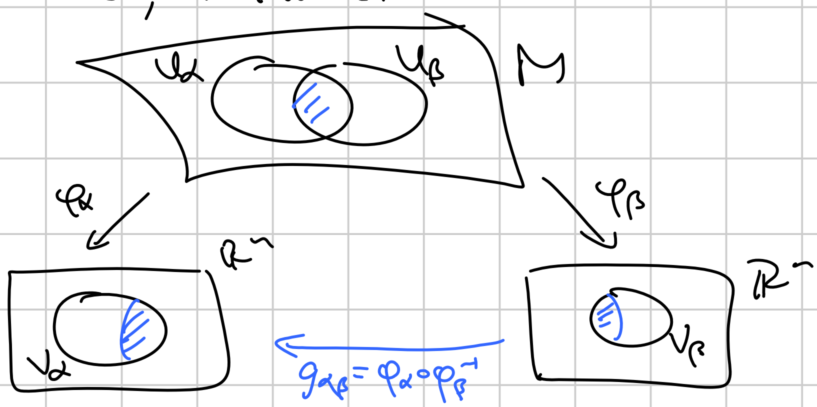
- text: Hatcher; Bott–Tu
- assignments: weekly HW, collected biweekly
  take-home final
- classes run through 12/4
- office hrs: Mondays 3–4, Wednesdays 2:30–4, and by appointment.

Various objects of study in topology.

→ $\boxed{\text{Topological spaces}}$ up to homeomorphism

→ $\boxed{\text{Topological manifolds}}$

- Hausdorff, 2nd countable
- Can cover w/ coord charts s.t. transition functions $g_{\alpha\beta}$ are continuous, cts inverse.



Study these up to homeomorphism.

→ $\boxed{\text{Smooth manifolds}}$

· Maps are smooth (∞-ly differentiable), smooth inverse.

Study up to diffeomorphism.

Smooth mfd

↓ note ∃ smooth mfds homeo but not diffeo; non-smoothable top mfds

Topological mfd

↓ note ∃ homology mfd not top mfd

Homology mfd

$H_*(X, X-\{x\}) = \begin{cases} \mathbb{Z}, & *=n \\ 0, & \text{otherwise} \end{cases}$

↓ note ≤ 3-mfds are triangulable; not 4

Δ-complex (simplicial complex)

↓

Topological space

deRham cohomology

∃ Poincaré duality

Simplicial (co)homology

$\shortparallel$

singular (co)homology

$\shortparallel$

Čech cohomology

Outline of course

- Singular cohomology, cup product, Poincaré duality (Hatcher ch. 3)
- differential forms, deRham cohomology, PD, Künneth Thm (Bott-Tu §1.1-5)
- Presheaves, Čech cohomology (BT §2)
- Spectral sequences, equiv. of cohomologies, Leray-Serre spectral sequence (BT §3)
- Vector bundles, Thom-isomorphism (BT §1.6).

Algebraic topology: associate algebraic objects to topological objects.

Category theory language – sometimes useful to adopt.

<u>Def</u> A <u>category</u> C is a set of <u>objects</u>, $\mathrm{Ob}\, C$,
and to any objects $A, B \in \mathrm{Ob}\, C$, a set of <u>morphisms</u> $\mathrm{Mor}(A,B)$ s.t.:

1. $\exists\, 1_A \in \mathrm{Mor}(A,A)$
2. $f \in \mathrm{Mor}(A,B),\ g \in \mathrm{Mor}(B,C) \Rightarrow g \circ f \in \mathrm{Mor}(A,C)$
3. associative
4. $1 \circ f = f \circ 1 = f$.

<u>Ex</u> (Set, set maps), (Grp, homomorphisms), (Ring, ring homoms),
(Ab = $\mathbb{Z}$-mod, homoms), (k-Vect, linear maps)
→ (Top, cts maps), (Smooth, smooth maps).

A <u>covariant functor</u> F from a category C to D
is a map $\mathrm{Ob}\, C \to \mathrm{Ob}\, D$ $\qquad$ $\mathrm{Mor}(A,B) \to \mathrm{Mor}(F(A),F(B))$

$A \mapsto F(A)$ $\qquad$ $f \mapsto F(f)$

s.t. $F(1_A) = 1_{F(A)}$ and $F(g) \circ F(f) = F(g \circ f)$.

ex: $\mathrm{Top} \xrightarrow{\pi_1} \mathrm{Grp}$, $\quad \mathrm{Top} \xrightarrow{H(\cdot, R)} R\text{-mod}$

A <u>contravariant</u> functor goes the other way.

ex: $\mathrm{Top} \xrightarrow{\text{functions}} k\text{-Vect}$.

The most "fundamental" functors in algebraic topology:

<u>Covariant</u>

$\mathrm{Top} \xrightarrow{\pi_1} \mathrm{Grp}$

$\xrightarrow{\pi_n} \mathrm{Ab}\ (n \geq 2),\ \mathrm{Set}\ (n=0)$

<u>Contravariant</u>

$\mathrm{Top} \xrightarrow{H^*} \mathrm{Ring}$ ←--- $H^*_{\mathrm{sing}}, \check{H}^*$

$\mathrm{Smooth} \xrightarrow{H^*_{\mathrm{DR}}} \mathrm{Ring}$

Cohomology

Start with chain complex of free $\mathbb{Z}$-modules (abelian groups):

$\boxed{\mathcal{C}}$ $$\cdots \to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \cdots \qquad \partial^2 = 0$$

ex: $C_n = C_n(X)$ singular chains.

$\longrightarrow$ Homology $\quad H_n(\mathcal{C}) = \ker(\partial : C_n \to C_{n-1}) / \operatorname{im}(\partial : C_{n+1} \to C_n)$.

Get other chain complexes from this:

Let G = abelian group (typically $\mathbb{Z}$; $\mathbb{Z}/n$; $\mathbb{Q}$; $\mathbb{R}$; $\mathbb{C}$).

$\boxed{\mathcal{C} \otimes G}$ $$\cdots \to C_{n+1} \otimes G \xrightarrow{\partial \otimes id} C_n \otimes G \xrightarrow{\partial \otimes id} C_{n-1} \otimes G \longrightarrow \cdots \qquad (\partial \otimes id)^2 = 0.$$

$\to$ Homology with G coefficients

$$H_n(\mathcal{C}; G) = \ker(\partial \otimes id : C_n \otimes G \to C_{n-1} \otimes G) / \operatorname{im}(\partial \otimes id : C_{n+1} \otimes G \to C_n \otimes G).$$

(note $H_n(\mathcal{C}; \mathbb{Z}) \cong H_n(\mathcal{C})$)

Or: we can apply the Hom functor.

$\boxed{\mathcal{C}^* = \operatorname{Hom}(\mathcal{C}, G)}$ $$\cdots \leftarrow \underset{\overset{\shortparallel}{C^{n+1}}}{\operatorname{Hom}(C_{n+1}, G)} \xleftarrow{\delta} \underset{\overset{\shortparallel}{C^{n}}}{\operatorname{Hom}(C_n, G)} \xleftarrow{\delta} \underset{\overset{\shortparallel}{C^{n-1}}}{\operatorname{Hom}(C_{n-1}, G)} \leftarrow \cdots$$

δ defined by $\quad (\delta\varphi)(c_{n+1}) = \varphi(\partial c_{n+1}) \qquad \varphi \in C^n,\ c_{n+1} \in C_{n+1}$

This is a complex: $\delta^2 = 0$.

$\to$ Cohomology with G coefficients

$$H^n(\mathcal{C}; G) = \ker(\delta : C^n \to C^{n+1}) / \operatorname{im}(\delta : C^{n-1} \to C^n).$$

$\boxed{\text{Note}}$ Homology is <u>covariant</u>: chain map $\varphi : \mathcal{C} \to \mathcal{C}'$ $\quad (\varphi : C_n \to C_n')$

induces $\quad \varphi_* : H_n(\mathcal{C}) \to H_n(\mathcal{C}')$

Cohomology is <u>contravariant</u>: $\quad \varphi^\# : \operatorname{Hom}(C_n', G) \to \operatorname{Hom}(C_n, G)$

$\rightsquigarrow \varphi^* : H^n(\mathcal{C}'; G) \to H^n(\mathcal{C}; G)$.

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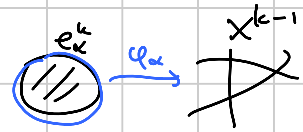
Refresher on cellular chain complex

Cell complex for X: $X = \cup$ cells. k-cell $\cong \overline{D^k}$

Build up inductively: $X^k = k$-skeleton.

$X^k = X^{k-1} \cup_{\varphi_\alpha} (\cup_\alpha e_\alpha^k)$, $e_\alpha^k = k$-cell, $\varphi_\alpha : S^{k-1} \to X^{k-1}$



Construct a complex $\mathcal{C}$ by

$$C_k = \mathbb{Z}\langle k\text{-cells}\rangle = \mathbb{Z}\langle e_1^k, \dots \rangle$$

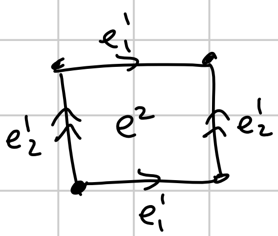
$\partial : C_k \to C_{k-1}$

$\partial(e_\alpha^k) = \sum_\beta n_{\alpha\beta} e_\beta^{k-1}$, $n_{\alpha\beta}$ = degree of $S^{k-1} \xrightarrow{\varphi_\alpha} X^{k-1} \xrightarrow{\text{collapse } X^{k-1} - \text{int}(e_\beta) \text{ to a point}} S^{k-1}$.

Then $H_n(\mathcal{C}) \cong H_n(X)$.

In practice: the boundary of a k-cell is $(k-1)$-dim'l, write this as a linear combination of $(k-1)$-cells. (Count with orientation!)

<u>Ex</u> $X = T^2$.



e^0
e_1^1, e_2^1
e^2

$\partial(e_1^1) = e^0 - e^0 = 0$

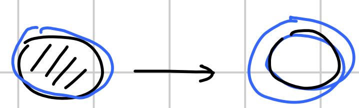
$\partial(e^2) = e_1^1 + e_2^1 - e_1^1 - e_2^1 = 0$.

	2		1		0
Cellular complex	$\mathbb{Z}$	$\xrightarrow{0}$	$\mathbb{Z}^2$	$\xrightarrow{0}$	$\mathbb{Z}$
homology $H_*(X)$	$\mathbb{Z}$		$\mathbb{Z}^2$		$\mathbb{Z}$
cohomology $H^*(X;\mathbb{Z})$	$\mathbb{Z}$		$\mathbb{Z}^2$		$\mathbb{Z}$

<u>Ex</u> $X = \mathbb{RP}^n$. $X^k = \mathbb{RP}^k$ One cell in dims $0, \dots, n$.

Attach k-cell to $\mathbb{RP}^{k-1}$ by $S^{k-1} \to \mathbb{RP}^{k-1}$

$\partial(e^k) = \begin{cases} 0 & k \text{ odd} \\ 2e^{k-1} & k \text{ even} \end{cases}$.



$k=2$

Ex Cellular chain complex for $\mathbb{RP}^n$ is

$$\mathcal{C}: \quad \mathbb{Z} \xrightarrow[2\,(\text{even})]{0\,(\text{odd})} \mathbb{Z} \longrightarrow \cdots \longrightarrow \mathbb{Z} \xrightarrow{0} \mathbb{Z} \xrightarrow{2} \mathbb{Z} \xrightarrow{0} \mathbb{Z}$$

	C_n	C_{n-1}		C_3	C_2	C_1	C_0	
	n	$n-1$		3	2	1	0	
$H_*(\mathbb{RP}^n)$:	0	$\mathbb{Z}/2$	$\cdots$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}$	n even
	$\mathbb{Z}$	0	$\cdots$	$\mathbb{Z}/2$	0	$\mathbb{Z}/2$	$\mathbb{Z}$	n odd
$\mathrm{Hom}(\mathcal{C},\mathbb{Z})$:	$\mathbb{Z}$	$\leftarrow \mathbb{Z} \leftarrow$	$\cdots$	$\leftarrow \mathbb{Z}$	$\xleftarrow{0} \mathbb{Z}$	$\xleftarrow{2} \mathbb{Z}$	$\xleftarrow{0} \mathbb{Z}$	
$H^*(\mathbb{RP}^n;\mathbb{Z})$:	$\mathbb{Z}/2$	0	$\cdots$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}$	n even
	$\mathbb{Z}$	$\mathbb{Z}/2$	$\cdots$	0	$\mathbb{Z}/2$	0	$\mathbb{Z}$	n odd
$H_*(\mathbb{RP}^n;\mathbb{Z}/2)$, $H^*(\mathbb{RP}^n;\mathbb{Z}/2)$:	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\cdots$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$	

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Universal Coefficient Theorem for cohomology/homology.

Miraculously, to calculate $H_*(\mathcal{C};G)$, $H^*(\mathcal{C};G)$ we only need $H_*(\mathcal{C})$ (!)

$\mathcal{C}$ = free chain complex, G = abelian group. Would like to say

$$H_n(\mathcal{C};G) \overset{?}{\cong} H_n(\mathcal{C}) \otimes G$$

$[c] \in H_n(\mathcal{C})$, $g \in G$ $\Rightarrow \partial c = 0 \Rightarrow (\partial \otimes \mathrm{id})(c \otimes g) = 0 \Rightarrow [c \otimes g] \in H_n(\mathcal{C};G)$

not $\cong$ in general! (see ex)

$$H^n(\mathcal{C};G) \overset{?}{\cong} \mathrm{Hom}(H_n(\mathcal{C}),G)$$

$[\varphi] \in H^n(\mathcal{C};G) \Rightarrow \varphi \in \mathrm{Hom}(C_n,G)$, $\delta\varphi = 0 \Rightarrow$ if $c = \partial c'$ then $\varphi(c) = \varphi(\partial c') = (\delta\varphi)(c') = 0$
$\Rightarrow \varphi: H_n(\mathcal{C}) \to G$.

UCT

$\mathcal{C}$ = free chain cx, G = abelian gp. Then $\exists$ exact sequences

$$0 \to H_n(\mathcal{C}) \otimes G \longrightarrow H_n(\mathcal{C};G) \longrightarrow \mathrm{Tor}(H_{n-1}(\mathcal{C}),G) \to 0$$

$$0 \leftarrow \mathrm{Hom}(H_n(\mathcal{C}),G) \longleftarrow H^n(\mathcal{C};G) \longleftarrow \mathrm{Ext}(H_{n-1}(\mathcal{C}),G) \leftarrow 0$$

that are natural and split (<u>not</u> naturally).

<u>Ext, Tor</u> G, H = abelian gp, free resolution $0 \to \underset{\text{relations}}{F_1} \to \underset{\text{generators}}{F_0} \to H \to 0$

(subgps of free abelian gps are free abelian)

$\rightsquigarrow$ complexes $\quad 0 \longrightarrow F_1 \otimes G \to \underbrace{F_0 \otimes G \longrightarrow H \otimes G \longrightarrow 0}_{\text{exact}}$

Tor(H,G) = homology here

$$0 \leftarrow \mathrm{Hom}(F_1,G) \leftarrow \overbrace{\mathrm{Hom}(F_0,G) \leftarrow \mathrm{Hom}(H,G) \leftarrow 0}^{\text{exact}}$$

Ext(H,G) = homology here

Handy guide: $\quad \mathrm{Ext}(\mathbb{Z},G)=0 \qquad \mathrm{Tor}(\mathbb{Z},G)=0$

$\mathrm{Ext}(\mathbb{Z}/n,G) = G/nG \qquad \mathrm{Tor}(\mathbb{Z}/n,G) = \ker(G \xrightarrow{n} G)$

$\mathrm{Ext}(H_1 \oplus H_2, G) = \mathrm{Ext}(H_1,G) \oplus \mathrm{Ext}(H_2,G) \qquad \mathrm{Tor}(H_1 \oplus H_2, G) = \mathrm{Tor}(H_1,G) \oplus \mathrm{Tor}(H_2,G)$

In particular: $\mathrm{Tor}(H,\mathbb{R}) = \mathrm{Ext}(H,\mathbb{R}) = 0$ for finitely generated H

$\Rightarrow$ <u>Cor 1</u> $\quad \boxed{H_n(\mathcal{C};\mathbb{R}) \cong H_n(\mathcal{C}) \otimes \mathbb{R}, \quad H^n(\mathcal{C};\mathbb{R}) \cong \mathrm{Hom}(H_n(\mathcal{C}),\mathbb{R})}$

(<u>not</u> true for general fields)

<u>Cor 2</u> If $H_n(\mathcal{C})$ has free part F_n, torsion part T_n, then $H^n(\mathcal{C};\mathbb{Z}) \cong F_n \oplus T_{n-1}$.

§2.8 ↑

Review of UCT for cohomology

G, H abelian gps, H = finitely generated.
Free resolution of H:

$$0 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow H \longrightarrow 0 \qquad \text{exact}$$

$$\Rightarrow \quad 0 \longleftarrow \mathrm{Hom}(F_1,G) \longleftarrow \mathrm{Hom}(F_0,G) \longleftarrow \mathrm{Hom}(H,G) \longleftarrow 0 \qquad \underline{\underline{\text{complex}}}$$

exact exact

$\mathrm{Ext}(H,G) :=$ homology here

$= \mathrm{coker}\,(\mathrm{Hom}(F_0,G) \to \mathrm{Hom}(F_1,G))$

$= \mathrm{Hom}(F_1,G) / \mathrm{im}\,(\mathrm{Hom}(F_0,G) \to \mathrm{Hom}(F_1,G))$.

exercise: $\mathrm{Ext}(\mathbb{Z},G) = 0$

$\mathrm{Ext}(\mathbb{Z}/n,G) = G/nG$

$\mathrm{Ext}(H_1 \oplus H_2, G) = \mathrm{Ext}(H_1,G) \oplus \mathrm{Ext}(H_2,G)$.

<u>UCT</u>: $\mathcal{C}$ = free chain cx, G = abelian gp. Then $\exists$ exact sequence

$$0 \to \mathrm{Ext}(H_{n-1}(\mathcal{C}),G) \longrightarrow H^n(\mathcal{C};G) \longrightarrow \mathrm{Hom}(H_n(\mathcal{C}),G) \longrightarrow 0$$

that is natural and splits (not naturally).

<u>Note</u>: if G = field then there's another thm, UCT with field coefficients:

$$\boxed{H^n(\mathcal{C};k) \cong \mathrm{Hom}_k(H_n(\mathcal{C};k),k)}.$$

(see HW).

Sketch of pf of UCT for cohomology

Complexes

$\mathcal{C}: \quad \cdots \to C_{n+1} \xrightarrow{\partial} C_n \xrightarrow{\partial} C_{n-1} \to \cdots$

$Z_n = \ker(\partial : C_n \to C_{n-1}) \qquad \mathcal{Z}: \quad \cdots \to Z_{n+1} \xrightarrow{0} Z_n \xrightarrow{0} Z_{n-1} \to \cdots$

$B_n = \operatorname{im}(\partial : C_{n+1} \to C_n) \qquad \mathcal{B}: \quad \cdots \to B_{n+1} \xrightarrow{0} B_n \xrightarrow{0} B_{n-1} \to \cdots$

$\Rightarrow$ SES of complexes

$$\begin{tikzcd} & \downarrow 0 & \downarrow \partial & \downarrow 0 & \\ 0 \arrow[r] & Z_* \arrow[r, "i"] \arrow[d, "0"] & C_* \arrow[r, "\partial"] \arrow[d, "\partial"] & B_{*-1} \arrow[r] \arrow[d, "\partial"] & 0 \\ & {} & {} & {} & \end{tikzcd}$$

$\Rightarrow$ SES $\quad 0 \leftarrow \operatorname{Hom}(Z_*, G) \xleftarrow{i^*} \operatorname{Hom}(C_*, G) \leftarrow \operatorname{Hom}(B_{*-1}, G) \leftarrow 0$

$\Rightarrow$ LES

$$\begin{tikzcd} & & \operatorname{Hom}(B_n, G) \arrow[dll, "i_n^*"'] \\ \operatorname{Hom}(Z_n, G) & H^n(\mathcal{C}; G) \arrow[l] & \operatorname{Hom}(B_{n-1}, G) \arrow[l] \arrow[dll, "i_{n-1}^*"] \\ \operatorname{Hom}(Z_{n-1}, G) & & \end{tikzcd}$$

$B_n \xrightarrow{i_n} Z_n$
$B_{n-1} \xrightarrow{i_{n-1}} Z_{n-1}$

$\Rightarrow$ SES $\quad 0 \to \operatorname{coker} i_{n-1}^* \to H^n(\mathcal{C}; G) \to \ker i_n^* \to 0$

Free resolution $\quad 0 \to B_n \xrightarrow{i_n} Z_n \to H_n(\mathcal{C}) \to 0$

$\Rightarrow 0 \leftarrow \underbrace{\operatorname{Ext}(H_n(\mathcal{C}), G)}_{\operatorname{coker} i_n^*} \leftarrow \operatorname{Hom}(B_n, G) \xleftarrow{i_n^*} \operatorname{Hom}(Z_n, G) \leftarrow \underbrace{\operatorname{Hom}(H_n(\mathcal{C}), G)}_{\ker i_n^*} \leftarrow 0$

$\Rightarrow$ SES

$$0 \to \operatorname{Ext}(H_{n-1}(\mathcal{C}), G) \to H^n(\mathcal{C}; G) \to \operatorname{Hom}(H_n(\mathcal{C}), G) \to 0.$$

UCT for cohom, field coefficients

k = field, $\mathcal{C}$ = free chain cx. Then $\exists$ natural isomorphism of vector spaces

$$\boxed{H^n(\mathcal{C}; k) \cong \operatorname{Hom}_k(H_n(\mathcal{C}; k), k)}.$$

Cor 3 If $f: \mathcal{C} \to \mathcal{C}'$ is a chain map inducing $\cong$ on homology then
$$f_*: H_n(\mathcal{C}; G) \to H_n(\mathcal{C}'; G)$$
$$f^*: H^n(\mathcal{C}'; G) \to H^n(\mathcal{C}; G) \quad \text{are } \cong.$$

Pf Use naturality and the five lemma. $\square$

So: say X = topological space. Let $\mathcal{C}$ be any chain complex computing the homology of X.
(eg $\mathcal{C}$ = singular chain cx, simplicial chain cx, cellular chain cx).

Def
$$H_n(X; G) = H_n(\mathcal{C}; G)$$
$$H^n(X; G) = H^n(\mathcal{C}; G).$$

These are well-defined and functorial:
$$f: X \to Y \rightsquigarrow \quad f_*: H_n(X; G) \to H_n(Y; G)$$
$$f^*: H^n(Y; G) \to H^n(X; G)$$

Notation: $\mathcal{C}: \quad \cdots \xrightarrow{\partial} C_n(X) \xrightarrow{\partial} \cdots$

$\mathrm{Hom}(\mathcal{C}, G): \quad \cdots \xleftarrow{\delta} C^n(X; G) \xleftarrow{\delta}$

$C^n(X;G) = \mathrm{Hom}(C_n(X), G)$

elements are <u>cochains</u>;
$\ker \delta$ = <u>cocycles</u>;
$\mathrm{im}\, \delta$ = <u>coboundaries</u>.

Let's see this explicitly for singular cohomology.

$C_n(X)$ gen'd by n-simplices $[v_0, v_1, \dots, v_n]$

$\varphi \in C^n(X; G) \Rightarrow \delta\varphi \in C^{n+1}(X; G) \quad (\delta\varphi)[v_0, \dots, v_{n+1}] = \varphi(\partial[v_0, \dots, v_{n+1}])$.

$f: X \to Y \rightsquigarrow$ map on chains $f_\# : C_n(X) \to C_n(Y)$

$[v_0, \dots, v_n] \longmapsto f_\#[v_0, \dots, v_n]$ (n-simplex in Y)

$f^\# : C^n(Y;G) \to C^n(X;G)$

$f^\# \varphi([v_0, \dots, v_n]) = \varphi(f_\#[v_0, \dots, v_n])$

$f_\# \partial = \partial f_\# \Rightarrow f_\#$ descends to $f_* : H_n(X) \to H_n(Y)$

$f^\# \delta = \delta f^\# \Rightarrow f^\#$ descends to $f^* : H^n(Y;G) \to H^n(X;G)$.

Cup Product

Maybe the main reason to prefer cohomology to homology: it has a ring structure.

We'll define for simplicial, singular cohomology.

so X = topological space. Also need coeffs in a <u>commutative ring</u> R.

$\varphi \in C^n(X;R)$ means for any simplex $[v_0, \dots, v_n] \to X$, get $\varphi([v_0, \dots, v_n]) \in R$.

For $\varphi \in C^k(X;R)$, $\psi \in C^\ell(X;R)$, define <u>$\varphi \smile \psi \in C^{k+\ell}(X;R)$</u> by

$$(\varphi \smile \psi)(\overbrace{[v_0, \dots, v_{k+\ell}]}^{\sigma}) = \varphi(\underbrace{\sigma|_{[v_0, \dots, v_k]}}_{\text{"front } k\text{-face"}}) \cdot \psi(\underbrace{\sigma|_{[v_k, \dots, v_{k+\ell}]}}_{\text{"back } \ell\text{-face"}}).$$

Prop $\delta(\varphi \cup \psi) = (\delta\varphi) \cup \psi + (-1)^k \varphi \cup (\delta\psi)$.

Pf $\sigma = [v_0, \dots, v_{k+\ell+1}]$.

$$
\begin{aligned}
\delta(\varphi\cup\psi)(\sigma) &= (\varphi\cup\psi)(\partial\sigma) \\
&= \sum_{0}^{k+\ell+1} (-1)^j (\varphi\cup\psi)([v_0, \dots, \hat{v}_j, \dots, v_{k+\ell+1}]) \\
&= \sum_{0}^{k} (-1)^j \varphi([v_0, \dots, \hat{v}_j, \dots, v_{k+1}])\,\psi([v_{k+1}, \dots, v_{k+\ell+1}]) \\
&\quad + \sum_{k+1}^{k+\ell+1} (-1)^j \varphi([v_0, \dots, v_k])\,\psi([v_k, \dots, \hat{v}_j, \dots, v_{k+\ell+1}]) \\
&= \varphi(\partial[v_0, \dots, v_{k+1}])\,\psi[v_{k+1}, \dots, v_{k+\ell+1}] \\
&\quad + (-1)^k \varphi([v_0, \dots, v_k])\,\psi(\partial[v_k, \dots, v_{k+\ell+1}]) \\
&= ((\delta\varphi)\cup\psi + (-1)^k \varphi\cup(\delta\psi))[v_0, \dots, v_{k+\ell+1}]. \qquad \square
\end{aligned}
$$

can replace by $k+1$

k

Cor $(\varphi, \psi) \mapsto \varphi \cup \psi$ descends to a map

$$\cup : \quad H^k(X;R) \otimes H^\ell(X;R) \longrightarrow H^{k+\ell}(X;R).$$

This gives $H^*(X;R) = \bigoplus_k H^k(X;R)$ the structure of a (sign) commutative ring, with unit if R has a unit: (in fact, R-algebra)

$\cup$ is associative, distributive, and sign commutative:

$$[\varphi] \cup [\psi] = (-1)^{k\ell} [\psi] \cup [\varphi] \qquad \text{(note NOT true on cochain level).}$$

Pf $\varphi \in C^k(X;R)$, $\psi \in C^\ell(X;R)$. If $\delta\varphi = \delta\psi = 0$ then $\delta(\varphi\cup\psi) = 0$.

If $\delta\varphi = 0$ and $\psi = \delta\psi'$ then $\varphi\cup\psi = \varphi\cup\delta\psi' = (-1)^k \delta(\varphi\cup\psi')$

If $\varphi = \delta\varphi'$ and $\delta\psi = 0$ then $\varphi\cup\psi = \delta\varphi'\cup\psi = \delta(\varphi'\cup\psi)$.

Assoc, dist true on cochain level.

Unit: if $1 \in R$ is unit then $\exists$ cocycle $\varphi \in C^0(X;R)$ with $\varphi([v_0]) = 1 \ \forall v_0 \in X$. $\delta\varphi([v_0, v_1]) = 0$ and $(\varphi \cup \psi)(\sigma) = (\psi \cup \varphi)(\sigma) = \psi(\sigma)$.

Commutativity: we'll outline; see pp. 216-7.

Given $\sigma : [v_0, \dots, v_n] \to X$ define $\bar{\sigma}$ by $[v_0, \dots, v_n] \to [v_n, \dots, v_0] \xrightarrow{\sigma} X$: reverse order of vertices in the simplex. This is a composition of $\frac{n(n+1)}{2}$ transpositions so it changes orientation by $(-1)^{\frac{n(n+1)}{2}}$. Define

$$\rho : C_n(X) \to C_n(X) : \qquad \rho(\sigma) = (-1)^{\frac{n(n+1)}{2}} \bar{\sigma}.$$

Lemma ρ = chain homotopic to id.

Then:
$$(\rho^*\varphi) \cup (\rho^*\psi)(\sigma) = (-1)^{\frac{k(k+1)}{2}} (-1)^{\frac{\ell(\ell+1)}{2}} \varphi(\sigma|[v_k, \dots, v_0]) \psi(\sigma|[v_{k+\ell}, \dots, v_k])$$
$$= (-1)^{k\ell} (-1)^{\frac{(k+\ell)(k+\ell+1)}{2}} \psi(\sigma|[v_{k+\ell}, \dots, v_k]) \varphi(\sigma|[v_k, \dots, v_0])$$
$$= (-1)^{k\ell} \rho^*(\psi \cup \varphi)(\sigma).$$

9/2 $\rho \sim \mathrm{id} \Rightarrow [\varphi] \cup [\psi] = (-1)^{k\ell} [\psi \cup \varphi]$. $\square$

Prop $f : X \to Y$ continuous map. Then $f^* : H^*(Y;R) \to H^*(X;R)$ preserves ring structure.

Pf Already true on cochain level:

$\varphi \in C^k(Y;R)$, $\psi \in C^\ell(X;R)$

$$f^\#(\varphi \cup \psi)(\underbrace{[v_0, \dots, v_{k+\ell}]}_{\sigma}) = (\varphi \cup \psi)(\overbrace{f_\# [v_0, \dots, v_{k+\ell}]}^{f\sigma})$$
$$= \varphi(f\sigma|[v_0, \dots, v_k]) \psi(f\sigma|[v_k, \dots, v_{k+\ell}])$$
$$= (f^\#\varphi)(\sigma|[v_0, \dots, v_k]) (f^\#\psi)(\sigma|[v_k, \dots, v_{k+\ell}])$$
$$= ((f^\#\varphi) \cup (f^\#\psi))(\sigma). \quad \square$$

Geometric Description of Cocycles

$X = \Delta$-complex. $\varphi \in C^k(X;\mathbb{Z})$.

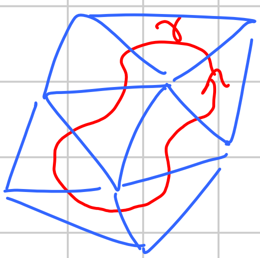
What does it mean that $\delta\varphi = 0$?

Toy model: X = Surface, $\varphi \in C^1(X;\mathbb{Z})$. φ associates an integer to each 1-simplex s.t.:

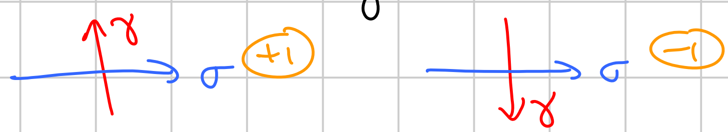
$$(\delta\varphi)(\sigma) = \varphi(\partial\sigma) = \varphi(\sigma_1) + \varphi(\sigma_2) + \varphi(\sigma_3) = 0$$
$\forall$ triangles σ.

One way to do this: take an oriented closed curve γ transverse to 1-simplices.

Then if σ = 1-simplex, define:



$\varphi(\sigma)$ = # of intersections of $\sigma \cap \gamma$ counted with sign:



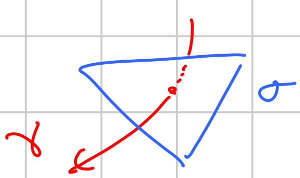
Then if σ = 2-simplex, $\varphi(\partial\sigma)$ = (# of times γ enters σ) − (# of times γ exits σ) $= 0$.

So φ = 1-cocycle. Call φ the Poincaré dual to γ:

$$\varphi = PD(\gamma).$$

Can generalize: X = top. mfd of dim n.

γ = $(n-k)$-cycle in X transverse to k-skeleton.

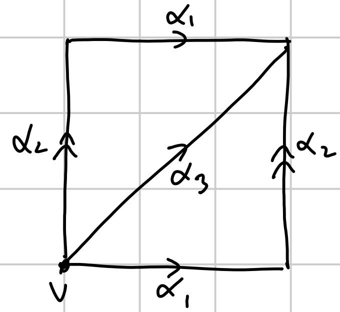


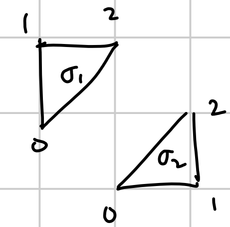
σ = k-simplex: define

$\varphi(\sigma)$ = # of intersections $\sigma \cap \gamma$.

$\rightsquigarrow \varphi \in C^k(X;\mathbb{Z})$, $\delta\varphi = 0$. $\underline{\varphi = PD(\gamma)}$.

Ex T^2.





$$\partial\sigma_1 = \alpha_1 + \alpha_2 - \alpha_3$$
$$\partial\sigma_2 = \alpha_1 + \alpha_2 - \alpha_3$$
$$\partial\alpha_i = 0.$$

$$\begin{array}{ccccc} C_2 & \longrightarrow & C_1 & \longrightarrow & C_0 \\ \| & & \| & & \| \\ \mathbb{Z}^2 & & \mathbb{Z}^3 & & \mathbb{Z} \\ \langle\sigma_1,\sigma_2\rangle & & \langle\alpha_1,\alpha_2,\alpha_3\rangle & & \langle v\rangle \end{array}$$

$H_2(T^2)\cong\mathbb{Z}$, $H_1(T^2)\cong\mathbb{Z}^2$, $H_0(T^2)\cong\mathbb{Z}$.

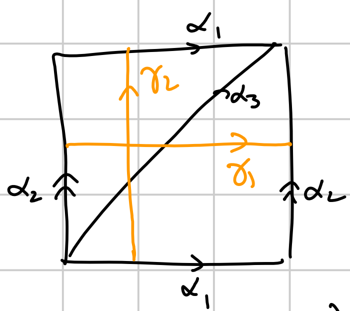
$$C^2 \xleftarrow{\begin{bmatrix}1 & 1 & -1\\ 1 & 1 & -1\end{bmatrix}} C^1 \xleftarrow{\begin{bmatrix}0\\0\\0\end{bmatrix}} C^0$$

UCT: $H^2(T^2)\cong\mathbb{Z}$, $H^1(T^2)\cong\mathbb{Z}^2$, $H^0(T^2)\cong\mathbb{Z}$.

- $H^0(T^2)$ generated by 1 where $1([v])=1$.
- $H^2(T^2)$ generated by $[\Theta]$ where $\Theta(\sigma_1)=0$, $\Theta(\sigma_2)=1$.

(Note: if $\varphi\in C^2$ with $\varphi(\sigma_1)=\varphi(\sigma_2)=c$ then $\varphi=\delta\psi$ for some $\psi\in C^1$:
define $\psi(\alpha_1)=c$, $\psi(\alpha_2)=\psi(\alpha_3)=0$. So $\begin{array}{ccc} C^2(T^2) & \longrightarrow & H^2(T^2)=\mathbb{Z}\langle\Theta\rangle \\ \varphi & \longmapsto & \varphi(\sigma_2)-\varphi(\sigma_1) \end{array}$.)

- How to picture $H^1(T^2)$? Use Poincaré duals.



$\varphi_1 = PD(\gamma_1)$, $\varphi_2 = PD(\gamma_2)$.

$\varphi_1(\alpha_1)=0$ $\quad\varphi_1(\alpha_2)=-1$ $\quad\varphi_1(\alpha_3)=-1$

$\varphi_2(\alpha_1)=1$ $\quad\varphi_2(\alpha_2)=0$ $\quad\varphi_2(\alpha_3)=1$

Note $\delta\varphi_1=\delta\varphi_2=0 \Rightarrow [\varphi_1],[\varphi_2]\in H^1(T^2;\mathbb{Z})$.

$[\varphi_1]\cup[\varphi_2]\in H^2(T^2;\mathbb{Z})$. What is it?

$(\varphi_1\cup\varphi_2)(\sigma_1) = (-1)(1)$ $\qquad (\varphi_2\cup\varphi_1)(\sigma_1)=(0)(0)$

$(\varphi_1\cup\varphi_2)(\sigma_2) = (0)(0)$ $\qquad (\varphi_2\cup\varphi_1)(\sigma_2)=(1)(-1)$

$\Rightarrow [\varphi_1\cup\varphi_2]=[\Theta]$ $\qquad \Rightarrow [\varphi_2\cup\varphi_1]=-[\Theta]$.

Alt: by UCT, $H^2(T^2;\mathbb{Z}) \xrightarrow{\cong} \mathrm{Hom}(H_2(T^2),\mathbb{Z}) \cong \mathbb{Z}$
identifies $[\Theta]$ with map sending $\sigma_2-\sigma_1 \mapsto 1$

So just calculate $(\varphi_1 \cup \varphi_2)(\sigma_2-\sigma_1) = 1$, $(\varphi_2\cup\varphi_1)(\sigma_2-\sigma_1) = -1$.

Note: since $[\Theta]$ generates H^2, $[\varphi_1], [\varphi_2] \neq 0 \in H^1$.
And in fact they generate H^1.

Full multiplication table for
$$H^*(T^2;\mathbb{Z}) \cong \mathbb{Z}\langle \underbrace{1}_{H^0}, \underbrace{[\varphi_1],[\varphi_2]}_{H^1}, \underbrace{[\Theta]}_{H^2}\rangle :$$

$\cup$	1	$[\varphi_1]$	$[\varphi_2]$	$[\Theta]$
1	1	$[\varphi_1]$	$[\varphi_2]$	$[\Theta]$
$[\varphi_1]$	$[\varphi_1]$	0	$[\Theta]$	0
$[\varphi_2]$	$[\varphi_2]$	$-[\Theta]$	0	0
$[\Theta]$	$[\Theta]$	0	0	0

note sign-commutativity;

Compare

$\cap$	γ_1	γ_2
γ_1	0	1
γ_2	-1	0

In fact, this is true in general:
if γ_1, γ_2 are closed curves in a connected oriented surface Σ, and $\varphi_1, \varphi_2 \in H^1(\Sigma;\mathbb{R})$ are PD to γ_1, γ_2, then
$$\varphi_1 \cup \varphi_2 = \#(\gamma_1 \cap \gamma_2)\,\Theta \qquad \Theta = \text{generator of } H^2(\Sigma;\mathbb{R}).$$

(This even holds for $\gamma_1=\gamma_2=\gamma$! Perturb $\gamma \rightsquigarrow \gamma'$ so that $\gamma \pitchfork \gamma'$.)

9/4 L

The ring $H^*(T^2;\mathbb{Z})$ is an example of an <u>exterior algebra</u>.

<u>Def</u> R = Comm ring with 1, M = R-module.

$T_R M$ = <u>tensor alg</u> of M over R

$= R \oplus M \oplus (M \otimes_R M) \oplus \cdots$ (generators as R-module: 1; $m_1 \otimes m_2 \otimes \cdots \otimes m_k$)

$\Lambda_R M$ = <u>exterior alg</u> of M over R

$= T_R M / I$

where I = ideal generated by $m \otimes m$, $m \in M$.

Write multiplication in $\Lambda_R M$ as $\wedge$; note $m \wedge m = 0 \Rightarrow$

$0 = (m_1 + m_2) \wedge (m_1 + m_2) = m_1 \wedge m_2 + m_2 \wedge m_1 \Rightarrow \boxed{m_1 \wedge m_2 = - m_2 \wedge m_1}$

The decomposition of $T_R M$ descends:

$$\Lambda_R M = \Lambda_R^0 M \oplus \Lambda_R^1 M \oplus \cdots \qquad \Lambda_R^r M \text{ gen'd by } m_1 \wedge \cdots \wedge m_r$$

and $\wedge : \Lambda_R^r M \otimes \Lambda_R^s M \to \Lambda_R^{r+s} M$.

Usual cases of interest:

- $R = \mathbb{Z}$, M = free $\mathbb{Z}$-module $= \mathbb{Z}\langle m_1, \ldots, m_n \rangle$.

  Then $\Lambda_R^r M$ gen'd by $m_{i_1} \wedge \cdots \wedge m_{i_r}$ $\quad i_1 < i_2 < \cdots < i_r$

  $\hookrightarrow$ rank $\binom{n}{r}$; $\quad \Lambda_R M$ has rank 2^r.

- $R = k$ field, M = k-vector space of dim n.

  Then $\Lambda_k^r M$ has $\dim_k = \binom{n}{r}$; $\Lambda_k M$ has dim 2^n.

Our case: $\quad H^*(T^2;\mathbb{Z}) \cong \Lambda_{\mathbb{Z}}^* M \qquad M = \mathbb{Z}\langle \varphi_1, \varphi_2 \rangle$

$$\begin{aligned} 1 &\longmapsto 1 \\ \varphi_1 &\longmapsto \varphi_1 \\ \varphi_2 &\longmapsto \varphi_2 \\ \Theta &\longmapsto \varphi_1 \wedge \varphi_2 . \end{aligned}$$

Another view: Künneth.

Künneth Theorem

Working through some examples:

$$\dim_k H_*(X\times Y;k) = \left(\dim_k H_*(X;k)\right)\left(\dim_k H_*(Y;k)\right).$$

Guess: $H_*(X\times Y;k) \cong H_*(X;k)\otimes H_*(Y;k)$.

In fact, often true for R = comm. ring.

Easier to state in cohomology.

$$\begin{tikzcd} & X\times Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd}$$

$\varphi \in H^k(X;R)$ $\rightsquigarrow$ $p_1^*\varphi \in H^k(X\times Y;R)$

$\psi \in H^\ell(Y;R)$ $\rightsquigarrow$ $p_2^*\varphi \in H^\ell(X\times Y;R)$

Def The <u>cross product</u> is

$$\times : \quad H^k(X;R)\otimes H^\ell(Y;R) \longrightarrow H^{k+\ell}(X\times Y;R).$$
$$\varphi \otimes \psi \longmapsto \varphi\times\psi = p_1^*\varphi \cup p_2^*\psi.$$

Sum over all k,ℓ to get

$$\boxed{\times : \ H^*(X;R)\otimes H^*(Y;R) \to H^*(X\times Y;R).}$$

Künneth: this map is often $\cong$.

As what?

- R-modules
- graded R-modules? RHS is graded: $H^* = \bigoplus_n H^n$.

  LHS? analogue of H^n should be $\bigoplus_{k+\ell=n} H^k(X;R)\otimes H^\ell(Y;R)$.

More generally: suppose $A^* = \bigoplus_n A^n$, $B^* = \bigoplus_m B^m$ are graded R-modules.

$$\rightsquigarrow \quad A^* \otimes B^* = \bigoplus_n \underbrace{\left(\bigoplus_{k+l=n} A^k \otimes B^l \right)}_{n^{th} \text{ graded piece of } A^* \otimes B^*}.$$

$$\begin{array}{c|ccc}
B^2 & A^0\otimes B^2 & A^1\otimes B^2 & A^2\otimes B^2 \\
B^1 & A^0\otimes B^1 & A^1\otimes B^1 & A^2\otimes B^1 \\
B^0 & A^0\otimes B^0 & A^1\otimes B^0 & A^2\otimes B^0 \\
\hline
 & A^0 & A^1 & A^2
\end{array}$$

Then $\times$ is a map of graded R-modules.

In fact, $\times$ is a map of graded R-algebras: it's a ring homomorphism.

How do we define multiplication on $H^*(X;R) \otimes H^*(Y;R)$?

$\varphi_1, \varphi_2 \in H^{k_1}(X;R), H^{k_2}$; $\psi_1, \psi_2 \in H^{l_1}(Y;R), H^{l_2}$.

Define $\quad (\varphi_1 \otimes \psi_1) \cdot (\varphi_2 \otimes \psi_2) = (-1)^{k_2 l_1} (\varphi_1 \cup \varphi_2) \otimes (\psi_1 \cup \psi_2)$.

Why? $\quad \times: \quad \varphi_1 \otimes \psi_1 \longmapsto (p_1^* \varphi_1) \cup (p_2^* \psi_1)$

$\varphi_2 \otimes \psi_2 \longmapsto (p_1^* \varphi_2) \cup (p_2^* \psi_2)$.

$$(\varphi_1 \otimes \psi_1) \cdot (\varphi_2 \otimes \psi_2) \overset{?}{\longmapsto} (p_1^* \varphi_1) \cup \underbrace{(p_2^* \psi_1)}_{\in H^{l_1}} \cup \underbrace{(p_1^* \varphi_2)}_{\in H^{k_2}} \cup (p_2^* \psi_2)$$

$$= (-1)^{k_2 l_1} \underbrace{(p_1^* \varphi_1) \cup (p_1^* \varphi_2)}_{p_1^*(\varphi_1 \cup \varphi_2)} \cup \underbrace{(p_2^* \psi_1) \cup (p_2^* \psi_2)}_{p_2^*(\psi_1 \cup \psi_2)}$$

$$= (-1)^{k_2 l_1} (\varphi_1 \cup \varphi_2) \times (\psi_1 \cup \psi_2). \quad \checkmark$$

Künneth Thm If $H^n(Y;R)$ is a finitely generated free R-module for all n, then

$$\times : \quad H^*(X;R) \otimes_R H^*(Y;R) \longrightarrow H^*(X\times Y;R)$$

is an isomorphism of graded R-algebras.

Note: if not: exact sequence

$$0 \to \bigoplus_{i+j=l} \left(H^i(X;R) \otimes H^j(Y;R)\right) \longrightarrow H^l(X\times Y;R)$$
$$\longrightarrow \bigoplus_{i+j=l} \operatorname{Tor}\left(H^{i+1}(X;R), H^j(Y;R)\right) \longrightarrow 0.$$

Ex $X = Y = S^1$, $X\times Y = T^2$.

$\varphi_1 \cup \varphi_1 = 0$

$H^*(X;\mathbb{Z})$ gen'd by $1 \in H^0$, $\varphi_1 \in H^1$; $H^*(Y;\mathbb{Z})$ gen'd by $1 \in H^0$, $\varphi_2 \in H^1$.

$H^*(X) \otimes H^*(Y)$ gen'd by: dim 0 $\quad 1\otimes 1$

dim 1 $\quad v_1 := \varphi_1 \otimes 1$, $v_2 := 1 \otimes \varphi_2$

dim 2 $\quad v_3 := \varphi_1 \otimes \varphi_2$.

$v_3 = v_1 \cup v_2$, $v_2 \cup v_1 = -v_1 \cup v_2$, $v_1 \cup v_1 = v_2 \cup v_2 = 0$.

$$H^*(T^2) \cong H^*(S^1) \otimes H^*(S^1) \cong \Lambda_{\mathbb{Z}} M \qquad M = \mathbb{Z}\langle v_1, v_2 \rangle.$$

Poincaré Duality

$H^*(T^n) \cong \Lambda_{\mathbb{Z}} M \quad M = \langle v_1, \dots, v_n \rangle$

k	0	1	2	$\cdots$	$n-1$	n
$\operatorname{rk} H^k(T^n;\mathbb{Z})$	1	n	$\binom{n}{2}$		$\binom{n}{n-1}$	1

This symmetry phenomenon is true in general and very useful.

Another ex: cellular cx for $\mathbb{CP}^n$:

$2n$	$2n-1$	$2n-2$		$\cdots$	1	0
$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\cdots$	0	$\mathbb{Z}$

k	0	1	2	$\cdots$	$2n-1$	$2n$
$\operatorname{rk} H^k(T^n;\mathbb{Z})$	1	0	1	$\cdots$	0	1

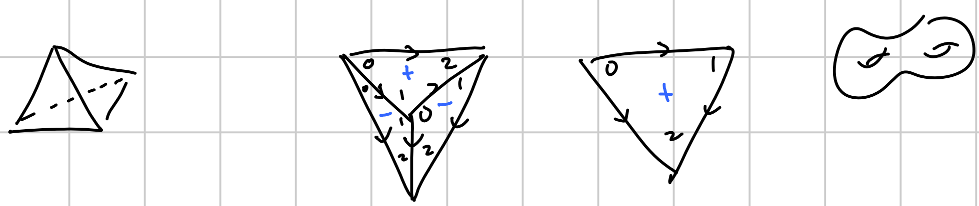
9/9

Orientations

M triangulable closed n-manifold, $M = \bigcup (n\text{-simplices})$, $\sigma_1, \dots, \sigma_k \in C_n(M)$. Suppose the ordering of vertices in σ_i and signs $\pm$ can be chosen so that $\sum \pm \partial \sigma_i = 0 \in C_{n-1}(X)$.

Def In this case, M is <u>orientable</u>.

<u>Ex.</u>



in HW.

If M is orientable, then $\sum \pm \sigma_i$ represents a class in $H_n(M)$, called a <u>fundamental class</u> $[M] \in H_n(M)$.

In fact, if M is connected, then (by PD) $H_n(M) \cong \mathbb{Z}$ and $[M]$ is a generator. There are two fundamental classes and a choice of one is an <u>orientation</u> of M.

Poincaré duality, version 1

If M is an orientable closed n-manifold, then for all k, $\exists$ isomorphism
$$H^k(M;G) \cong H_{n-k}(M;G)$$
for any abelian group G.

Rmks 1. Note $0 \leq k \leq n$. $\quad H^0(M;\mathbb{Z}) \cong H_n(M;\mathbb{Z}) = H_n(M)$.

Recall $H^k(M;\mathbb{Z}) \cong (\text{free part of } H_k(M)) \oplus (\text{torsion part of } H_{k-1}(M))$

So if M is connected, $\mathbb{Z} \cong H_0(M) \cong H^0(M;\mathbb{Z}) \cong H_n(M;\mathbb{Z})$.

2. If M isn't orientable, then PD still holds but only for $G = \mathbb{Z}/2$:
$$H^k(M;\mathbb{Z}/2) \cong H_{n-k}(M;\mathbb{Z}/2).$$

Ex $\mathbb{RP}^2$.

	2	1	0		2	1	0
$H_*(\cdot;\mathbb{Z})$:	0	$\mathbb{Z}/2$	$\mathbb{Z}$	$H_*(\cdot;\mathbb{Z}/2)$:	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$
$H^*(\cdot;\mathbb{Z})$:	$\mathbb{Z}/2$	0	$\mathbb{Z}$	$H^*(\cdot;\mathbb{Z}/2)$:	$\mathbb{Z}/2$	$\mathbb{Z}/2$	$\mathbb{Z}/2$

3. Betti numbers.
$$b_k := \text{free rank}\,(H^k(M;\mathbb{Z})) \overset{\text{UCT}}{=} \text{free rank}\,(H_k(M))$$
So PD $\Rightarrow$ $\boxed{b_k = b_{n-k} \quad \forall k}$.

There's a nice way to explicitly write down the isomorphism when G = comm ring.

Def For a space X and a comm ring R, define the <u>cap product</u>

$$\frown : C_k(X;R) \otimes C^\ell(X;R) \longrightarrow C_{k-\ell}(X;R) \qquad \text{for } k \geq \ell$$
$$\sigma \otimes \varphi \longmapsto \sigma \frown \varphi = \varphi(\sigma|[v_0,\dots,v_\ell])\,\sigma|[v_\ell,\dots,v_k].$$

Prop $\partial(\sigma \frown \varphi) = (-1)^{\ell}\big((\partial\sigma)\frown\varphi - \sigma\frown(\delta\varphi)\big)$.

Pf
$$\begin{aligned}(\partial\sigma)\frown\varphi &= \Big(\sum_j (-1)^j \sigma|[v_0,\dots,\hat{v}_j,\dots,v_k]\Big)\frown\varphi \\
&= \underbrace{\sum_{j\leq \ell \to \ell+1} (-1)^j \varphi(\sigma|[v_0,\dots,\hat{v}_j,\dots,v_{\ell+1}])}_{\leadsto \varphi(\partial\sigma|[v_0\cdots v_{\ell+1}]) = (\delta\varphi)(\sigma|\dots)} \sigma|[v_{\ell+1},\dots,v_k] \\
&\quad + \sum_{j\geq \ell+1 \to \ell} (-1)^j \varphi(\sigma|[v_0,\dots,v_\ell])\, \sigma|[v_\ell,\dots,\hat{v}_j,\dots,v_k] \\
&= \sigma\frown\delta\varphi + (-1)^{\ell} \sum_{j\geq 0} (-1)^j \varphi(\sigma|[v_0,\dots,v_\ell])\, \sigma|[v_\ell,\dots,\hat{v}_{\ell+j},\dots,v_k] \\
&= \sigma\frown\delta\varphi + (-1)^{\ell}\partial(\sigma\frown\varphi). \qquad \square\end{aligned}$$

Cor $\frown$ is well-defined: $H_k(X;R)\otimes H^{\ell}(X;R)\to H_{k-\ell}(X;R)$.

Poincaré duality, version 2

If M is an orientable closed n-manifold, let $[M]\in H_n(M)$ be a fundamental class for M, and let R be a commutative ring. View $[M]\in H_n(M;R)$; then the map

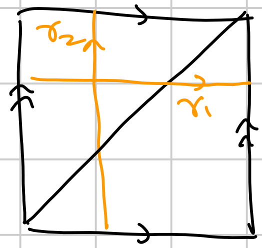
$$\begin{aligned} H^{\ell}(M;R) &\longrightarrow H_{n-\ell}(M;R) \\ \varphi &\longmapsto [M]\frown\varphi \end{aligned}$$

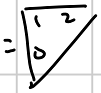
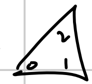
is an R-module isomorphism.

If M is nonorientable, then $[M]\in H_n(M;\mathbb{Z}/2)$ still exists, and we have

$$H^{\ell}(M;\mathbb{Z}/2)\xrightarrow{\cong} H_{n-\ell}(M;\mathbb{Z}/2). \qquad \text{(same map)}$$

Ex $M = T^2$.



$\sigma_1 =$  $\sigma_2 =$ 

$[T^2] = \sigma_2 - \sigma_1$.

$\varphi_1 = PD(\gamma_1)$, $\varphi_2 = PD(\gamma_2)$.

$$[T] \cap \varphi_1 = \varphi_1(\sigma_2|[v_0,v_1])\,\sigma_2|[v_1,v_2] - \varphi_1(\sigma_1|[v_0,v_1])\,\sigma_1|[v_1,v_2]$$
$$= (0) - (-1)\,\sigma_1|[v_1,v_2]$$
$$= \gamma_1 \text{ in homology.}$$

$[T] \cap \varphi_2 = (1)\,\sigma_2|[v_1,v_2] - (0) = \gamma_2$ in homology.

So $[T]\cap : \quad H^1(T^2;\mathbb{Z}) \longrightarrow H_1(T^2)$ (the PD map)

sends $\varphi_1 \longmapsto \gamma_1$, $\varphi_2 \longmapsto \gamma_2$.

Another view. Recall from UCT $H^n(M;R) \longrightarrow \mathrm{Hom}(H_n(M),R)$.

So the cup product gives a map

$$\cup : H^\ell(M;R) \otimes H^{n-\ell}(M;R) \longrightarrow H^n(M;R) \longrightarrow R.$$
$$\varphi \longmapsto \varphi([M])$$

Now assume either $R=$ field, or $R=\mathbb{Z}$ and no torsion in $H_*(M)$.

UCT for fields: $H^\ell(M;k) \cong \mathrm{Hom}(H_\ell(M;k),k)$

UCT: $0 \to \underset{=\,0}{\mathrm{Ext}(H_{\ell-1}(M),\mathbb{Z})} \to H^\ell(M;\mathbb{Z}) \to \mathrm{Hom}(H_\ell(M),\mathbb{Z}) \to 0$

In either case we have

$$H^\ell(M;R) \xrightarrow[\text{UCT}]{\cong} \mathrm{Hom}(H_\ell(M;R),R) \xrightarrow[\text{PD}]{\cong} \mathrm{Hom}(H^{n-\ell}(X;R),R).$$

$$\varphi \longmapsto (\sigma \mapsto \varphi(\sigma)) \longmapsto (\psi \mapsto \varphi([M]\cap\psi) = (\psi\cup\varphi)[M]).$$

Def A bilinear map $A \otimes_R B \to R$ is <u>nonsingular</u> if the induced maps $A \to \mathrm{Hom}(B,R)$, $B \to \mathrm{Hom}(A,R)$ are $\cong$.

Poincaré duality, version 3

M orientable closed n-mfd, R = field or $R = \mathbb{Z}$ and quotient out by torsion. Then

$$\cup : H^\ell(M;R) \otimes H^{n-\ell}(M;R) \longrightarrow R$$

9/11 is nonsingular.

Nice application: $H^*(\mathbb{C}P^n)$.

<u>Prop</u> $H^*(\mathbb{C}P^n;\mathbb{Z}) \underset{\text{as a ring.}}{\cong} \mathbb{Z}[\alpha]/(\alpha^{n+1})$ $\qquad |\alpha| = 2$ $(\alpha \in H^2)$.

<u>Pf</u> By induction on n.

$i : \mathbb{C}P^{n-1} \hookrightarrow \mathbb{C}P^n \rightsquigarrow i^* : H^*(\mathbb{C}P^n;\mathbb{Z}) \to H^*(\mathbb{C}P^{n-1};\mathbb{Z})$,

isomorphism for $* \le 2n-2$.

$$\begin{array}{lcccccc}
 & 0 & 2 & 4 & \cdots & 2n-2 & 2n \\
H^*(\mathbb{C}P^{n-1}) & \langle 1 \rangle & \langle \beta \rangle & \langle \beta^2 \rangle & \cdots & \langle \beta^{n-1} \rangle & — \\
 & \cong\uparrow & \cong\uparrow & \cong\uparrow & & \cong\uparrow & \\
H^*(\mathbb{C}P^n) & \langle 1 \rangle & \langle \alpha \rangle & \langle \alpha^2 \rangle & \cdots & \langle \alpha^{n-1} \rangle & \langle x \rangle
\end{array}$$

$\beta^{n-1} \neq 0$.

Let $\alpha \in H^2(\mathbb{C}P^n)$ satisfy $i^*\alpha = \beta$. Then $i^*(\alpha^{n-1}) = \beta^{n-1} \neq 0$

so $H^{2\ell}(\mathbb{C}P^n)$ is gen'd by α^ℓ, $\ell \le n-1$.

Say $H^{2n}(\mathbb{C}P^n)$ is gen'd by x. Then

$$\alpha \cup \alpha^{n-1} = m x \text{ for some } m.$$

Nonsingularity: $\quad H^2 \longrightarrow \mathrm{Hom}(H^{2n-2},\mathbb{Z}) \cong \mathbb{Z}$ $\quad$ is an isom.

$$\alpha \longmapsto (\alpha^{n-1} \mapsto mx) \longmapsto m$$

$\Rightarrow m = \pm 1$. In either case, we get $\alpha \cup \alpha^{n-1} = \alpha^n$ generates H^{2n}.

$\square$

Note Same argument shows $H^*(\mathbb{RP}^n;\mathbb{Z}/2)\cong\mathbb{Z}/2[\alpha]/(\alpha^{n+1})$, $\alpha\in H^1$.

Interlude: Borsuk-Ulam Theorem

From HW, consequence of $H^*(\mathbb{RP}^n;\mathbb{Z}/2)$:

Prop $\nexists$ continuous $f: S^n\to S^{n-1}$ with $f(x)=-f(-x)$ $\forall x\in S^n$.

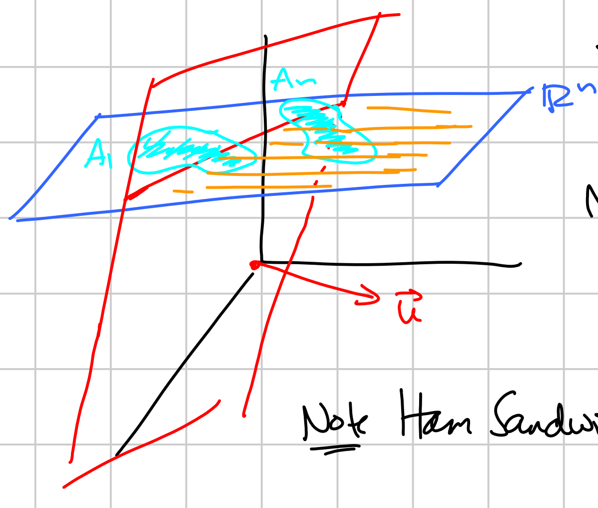
Borsuk-Ulam Thm For any continuous $g: S^n\to\mathbb{R}^n$, $\exists x\in S^n$ such that $g(x)=g(-x)$.

Pf Otherwise define $f: S^n\to S^{n-1}$ by $f(x)=\dfrac{g(x)-g(-x)}{\|g(x)-g(-x)\|}$. $\square$

Cor There are two antipodal points on Earth with same temperature+pressure.

Cor (Ham Sandwich Thm)
Let $A_1,\dots,A_n\subset\mathbb{R}^n$ be bounded measurable sets. Then there exists a hyperplane that divides each A_i into piece of equal measure.

Pf Place $\mathbb{R}^n=\mathbb{R}^n\times\{1\}\subset\mathbb{R}^{n+1}$. For $\vec{u}\in S^n\subset\mathbb{R}^{n+1}$ unit vector, let $\Pi_{\vec{u}}$ = hyperplane through 0 $\perp$ to $\vec{u}$. Let $g_i(\vec{u})$ = measure of the part of A_i on the same side of $\Pi_{\vec{u}}$ as $\vec{u}$.



Now apply Borsuk-Ulam to $g=(g_1,\dots,g_n): S^n\to\mathbb{R}^n$. $\square$

Note Ham Sandwich Thm nontrivial even for $n=2$ and A_1,A_2 = triangle!

Application: What this means for 4-manifolds

X = connected, simply connected, orientable closed topological 4-manifold.

HW: then the only interesting cohomology with $\mathbb{Z}$ coeffs is $H^2(X;\mathbb{Z})$. (torsion-free)

Cup product gives a bilinear form (intersection form)

$$Q: H^2(X;\mathbb{Z}) \otimes H^2(X;\mathbb{Z}) \longrightarrow H^4(X;\mathbb{Z}) \overset{PD}{\cong} \mathbb{Z}$$

Q is symmetric, bilinear, represented by an integer matrix M.

PD $\Rightarrow$ Q = nonsingular $\Rightarrow$ $\det M = \pm 1$ $\Rightarrow$ Q is <u>unimodular</u>.

<u>Thm</u> (Freedman, ~82)

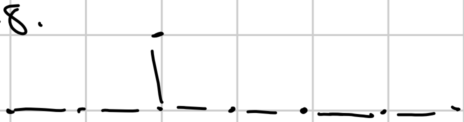
For any symmetric unimodular bilinear form Q on $\mathbb{Z}$, $\exists$ s.c. topological closed 4-mfd whose intersection form is Q.

<u>Thm</u> (Donaldson ~83)

If Q = intersection form for s.c smooth 4-mfd and Q = definite then Q = diagonalizable.

$\Rightarrow \exists$ symm unimodular bilinear forms Q st $\nexists$ sc <u>smooth</u> 4-mfd with that intersection form.

Ex: E_8.



$$\begin{bmatrix} 2 & 0 & -1 & & & & & \\ 0 & 2 & 0 & -1 & & & & \\ -1 & 0 & 2 & -1 & & & & \\ & -1 & -1 & 2 & -1 & & & \\ & & & -1 & 2 & -1 & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{bmatrix}$$

definite, not diag.

9/16 <u>Cor</u> $\exists$ topl 4-manifolds with <u>no</u> smooth structure!

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